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NONLINEARLY EQUIVALENT REPRESENTATIONS OF QUATERNIONIC 2-GROUPS

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ABSTRACT. We construct new examples of nonlinearly equivalent finite-dimensional real linear representations of quaternionic 2-groups, which cannot be obtained from equivalent representations of cyclic groups by induction and composition techniques.

INTRODUCTION

Let G be a finite group and V a finite-dimensional real vector space. A linear representation of G in V is a homomorphism $\rho: G \to \mathrm{GL}(V)$, where $\mathrm{GL}(V)$ is the general linear group of V. Two representations ρ_1 and ρ_2 of G in $\mathbf{R}^n$ are said to be linearly (topologically, resp.) equivalent if there exists a linear isomorphism (homeomorphism, resp.) $\phi: \mathbf{R}^n \to \mathbf{R}^n$ such that $\phi.\rho_1(g) = \rho_2(g).\phi$, for any $g \in G$.

All known examples of nonlinearly equivalent representations of general finite groups were obtained by standard group representation methods, namely, composition and induction, from the examples for cyclic groups $\mathbf{Z}_{4k}$, $k \geq 2$ [CS3], which in turn were obtained by Cappell and Shaneson by studying geometrically lens spaces [CS2]. As these methods give a (weak) stable topological classification of representations, the question has arisen (see the Problem List of the Boulder Transformation Groups Conference, 1983), if these are indeed all the topologically equivalent representations. Here we show that there are further new examples for nonabelian groups by considering more general space-forms than lens spaces. [1]

Let $\mathbf{H}_{2^{r+2}}$ be the quaternionic group of order 2^{r+2}, $r \geq 2$,

$$\mathbf{H}_{2^{r+2}} = \{x, y | x^2 = y^{2^r}, xyx^{-1} = y^{-1}\}$$

and $\phi_1, \psi_1, \theta_k, k$ odd, the real representations of $\mathbf{H}_{2^{r+2}}$ described below. ϕ_1 and ψ_1 are 1-dimensional representations given on generators by

$$\phi_1(x) = -1, \quad \phi_1(y) = -1,$$
$$\psi_1(x) = 1, \quad \psi_1(y) = -1.$$

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[1] Recently, new example for cyclic groups were obtained in [CSSW] by different techniques.

θ_k, k odd, is the 4-dimensional representation defined by

$$\theta_k(x) = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \qquad \theta_k(y) = \begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_{-k} \end{bmatrix}$$

where I is the 2×2 identity matrix and σ_j is the 2×2 rotation matrix by the angle $\pi.j/2^r$.

Let $\mathbf{Z}_{2m}$, $m = 2^r$, be the index two cyclic subgroup of $\mathbf{H}_{2^{r+2}}$ generated by y. The representations θ_k and $\psi_1 + \phi_1$ of $\mathbf{H}_{2^{r+2}}$ are induced (see e.g. [Se]) by the representations σ_k and δ_{-1} of $\mathbf{Z}_{2m}$, respectively, where δ_{-1} is the unique nontrivial 1-dimensional representation of $\mathbf{Z}_{2m}$ and σ_k is defined on the generator y as above. By [CS2], $4\sigma_k + \delta_{-1}$ and $4\sigma_{k+m} + \delta_{-1}$ are topologically equivalent; since nonlinear similarity is preserved under induction [CS3], it follows that

$$4\theta_k + \psi_1 + \theta_1 \sim 4\theta_{k+m} + \psi_1 + \phi_1 \tag{0.1}$$

where $\sim$ means topologically equivalent.

Our main result, stated below, shows that we can cancel either ϕ_1 or ψ_1 in (0.1). We remark that this result is the best possible, in that we cannot proceed further to cancel both ϕ_1 and ψ_1, since it is well known that, for free representations, topological and linear equivalence are the same. This improves the stable range for the nonlinear equivalence of $4\theta_k$ and $4\theta_{k+m}$ and consequently, for representations of finite groups obtained from θ_k and $4\theta_{k+m}$ by induction.

Theorem. $4\theta_k + \psi_1 \sim 4\theta_{k+m} + \psi_1$ *and* $4\theta_k + \phi_1 \sim 4\theta_{k+m} + \phi_1$.

As the proofs of the above statements are almost identical, we restrict our discussion to the first one, for which we apply the methods of [CS3].

This work is divided in three parts. §1 contains the basic material concerning quaternionic space forms and the computation of the normal invariant of some homotopy equivalences between these spaces. §2 provides the material necessary for the determination of some surgery obstructions for quaternionic groups that arise in our applications. The last section is devoted to the proof of the theorem stated above.

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1. Quaternionic space-forms

1.1. Preliminaries. Let $\mathbf{H}_{4m}$ be the quaternionic group of order $4m$,

$$\mathbf{H}_{4m} = \{x, y | x^2 = y^m, \; xyx^{-1} = y^{-1}\}. \tag{1.1.1}$$

Let $O(n)$ denote the group of orthogonal transformations of $\mathbf{R}^n$ and β_k the 2×2 rotation matrix by the angle $\pi.k/m$. From classical representation theory (see e.g. [Se]), any irreducible real representation of $\mathbf{H}_{4m}$, m even, is linearly equivalent to one of the following:

(1.1.2)

(a)
$$\psi_i : \mathbf{H}_{4m} \to O(1),$$
$$\psi_i(x) = 1, \qquad \psi_i(y) = (-1)^i, \qquad i = 0, 1;$$

(b)
$$\phi_i : \mathbf{H}_{4m} \to O(1),$$
$$\phi_i(x) = -1, \qquad \phi_i(y) = (-1)^i, \qquad i = 0, 1;$$

(c)
$$\delta_k : \mathbf{H}_{4m} \to O(2), \quad k \text{ even}, \ k \notin m.\mathbf{Z},$$
$$\delta_k(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \delta_k(y) = \beta_k;$$

(d)
$$\theta_k : \mathbf{H}_{4m} \to O(4), \quad k \text{ odd},$$
$$\theta_k(x) = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \qquad \theta_k(y) = \begin{bmatrix} \beta_k & 0 \\ 0 & \beta_{-k} \end{bmatrix}$$

where I is the 2×2 identity matrix.

Remark. θ_{k_1} and θ_{k_2} (δ_{k_2} and δ_{k_1}, resp.) are linearly equivalent if and only if $k_1 + k_2 \equiv 0 \pmod{2m}$ or $k_1 - k_2 \equiv 0 \pmod{2m}$.

Given integers k_i, $i = 1, \ldots, n$ and m even, $(k_i, 2m) = 1$, the restriction of the direct sum representation $\theta_{k_1} + \cdots + \theta_{k_n} : \mathbf{H}_{4m} \to O(4n)$ to the unit sphere $\mathbf{S}^{4n-1}$ of the representation space $\mathbf{R}^{4n}$ determines a free action of $\mathbf{H}_{4m}$ on $\mathbf{S}^{4n-1}$. The *linear quaternionic space form* $Q_{4m}(k_1, \ldots, k_n)$ is the quotient space $\mathbf{S}^{4n-1}/\mathbf{H}_{4m}$, which is an orientable $(4n-1)$-manifold, since each representation θ_{k_i} factors through $SO(4)$.

The integral homology groups of $Q_{4m}(k_1, \ldots, k_n)$ are given by

$$(1.1.3) \qquad H_i = \begin{cases} \mathbf{Z}, & i = 0 \text{ or } 4n-1, \\ \mathbf{Z}_2 + \mathbf{Z}_2, & i \equiv 1 \pmod 4, 0 < i < 4n-1, \\ \mathbf{Z}_{4m}, & i \equiv 3 \pmod 4, 0 < i < 4n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Recall that a *polarization* of $Q = Q_{4m}(k_1, \ldots, k_n)$ is given by a choice of orientation of Q and an isomorphism $\imath : \pi_1(Q) \to \mathbf{H}_{4m}$ (we are fixing basepoints in $\mathbf{S}^{4n-1}$ and in Q, compatibly). Similarly, we call an *outer polarization*, a choice of orientation together with an outer isomorphism of $\pi_1(Q)$ and $\mathbf{H}_{4m}$, i.e., an equivalence class of isomorphisms $\imath : \pi_1(Q) \to \mathbf{H}_{4m}$, where two such are said to be equivalent if they differ by an inner automorphism of $\mathbf{H}_{4m}$.

Hereforth, we shall fix the following (outer) polarization of Q:

(a) the orientation of Q induced by the usual orientation of $\mathbf{S}^{4n-1}$.

(b) The (class of the) standard identification of $\pi_1(Q)$ with the group of covering transformations.

Let $Q = Q_{4m}(k_1, \ldots, k_n)$ and $Q' = Q_{4m}(k'_1, \ldots, k'_n)$.

Proposition 1.1.4. *There exists a unique (unbased) based homotopy class of (outer) polarization preserving homotopy equivalences* $f: Q \to Q'$, *if* $k_1^2 \cdots k_n^2 \equiv k_1'^2 \cdots k_n'^2 \pmod{4m}$.

Proof. Let q_i $(i = 1, \ldots, n)$ be integers such that $q_i.k_i \equiv 1 \pmod{2m}$ and $h_i: S^1 \to S^1$ be given by $h_i(z) = z^{q_i.k_i}$. Let $\tilde{g}_i = h_i * h_i: S^3 \to S^3$ be the join of two copies of h_i and $\tilde{g}: S^{4n-1} \to S^{4n-1}$ be defined by $\tilde{g} = \tilde{g}_1 * \cdots * \tilde{g}_n$. One can easily check that $\tilde{g}$ covers a map $g: Q \to Q'$ of degree $d_0 = q_1^2 \cdots q_n^2.k_1^2 \cdots k_n^2$ inducing the "right" outer isomorphism on π_1. By hypothesis, $d_0 = 1 - 4m.l$, for some $l \in \mathbf{Z}$. Assume that S^{4n-1} is given a "suitable" triangulation compatible with the action $\theta = \theta_{k_1} + \cdots + \theta_{k_n}$ of $\mathbf{H}_{4m}$. In a top dimensional cell, we replace $\tilde{g}$ by a map of degree l and do the same on its θ-translates; this produces a map $\tilde{f}$ which covers a degree 1 map $f: Q \to Q'$. After a free homotopy, we can assume that f induces the required isomorphism on π_1 (this is not necessary in the unbased case).

By the same argument, we construct $\bar{f}: Q' \to Q$ satisfying analogous properties. Applying Lemma 2.1 of [Ma] to $f.\bar{f}$ and $\bar{f}.f$, we conclude that f is a (outer) polarization-preserving homotopy equivalence.

Uniqueness also follows from Lemma 2.1 of [Ma], since any two such maps differ by a (outer) polarized self-equivalence of Q'.

Remark. The condition stated above is also necessary; this reflects the fact that $k_1^2 \cdots k_n^2 \pmod{4m}$ is determined by the first k-invariant of Q, which in turn classifies polarized homotopy types (see e.g. [Ma, W7]). A more direct argument can be found in [Mi], which contains a more complete description of (based) homotopy classes of maps between Q and Q' (in the style of [Co], for lens spaces). This is omitted here since the above proposition suffices for our purposes.

A construction similar to the one in the proof of (1.1.4) yields

Proposition 1.1.5. *There exists an orientation preserving diffeomorphism* $f: Q \to Q'$ *if there are numbers* $a \in \mathbf{Z}$ *and* $\varepsilon_i \in \{\pm 1\}$, *such that* $(k_1, \ldots, k_n)$ *is a permutation of* $(\varepsilon_1 a k_1', \ldots, \varepsilon_n a k_n') \pmod{2m}$.

Remark. If $a \equiv 1 \pmod{2m}$, f can be taken to be polarization preserving. In general, however, such diffeomorphisms do not exist.

1.2. Normally cobordant quaternionic space-forms

Let $m, k_1, \ldots, k_n, t_1, \ldots, t_s$ be integers such that m is even, $(k_i, 2m) = 1$ and $(t_j, 2m) = 1$.

Let

$$Q = Q_{4m}(k_1, k_1, k_2, k_2, \ldots, k_n, k_n, t_1, t_2, \ldots, t_s)$$

and

$$Q' = Q_{4m}(k_1 + m, k_1 + m, \ldots, k_n + m, k_n + m, t_1, \ldots, t_s)$$

be the $(8n+4s-1)$-dimensional spherical space-forms defined in section 1.1. By Proposition (1.1.4), there exists a polarization preserving homotopy equivalence $f: Q \to Q'$, since the criterion obtained there can be easily checked in this case, as it reduces to

$$k_1^4 \cdots k_n^4 . t_1^2 \cdots t_s^2 \equiv (k_1 + m)^4 \cdots (k_n + m)^4 . t_1^2 \cdots t_s^2 \pmod{4m}.$$

Let G/Top be the classifying space for fibre homotopy trivialization of topological Euclidean space bundles, $[Q'; G/\mathrm{Top}]$ the set of homotopy classes of maps from Q' into G/Top and $\eta(f) \in [Q'; G/\mathrm{Top}]$, the topological normal invariant of f.

Proposition 1.2.1. *The normal invariant $\eta(f)$ of f is zero, i.e., f is topologically normally cobordant to the identity map of Q'.*

Before proceeding to the proof of this proposition, we prove a lemma of a more technical nature which will be used in the argument.

Let

$$Q = Q_{4m}(k, k, t_1, \ldots, t_s),$$
$$Q' = Q_{4m}(k+m, k+m, t_1, \ldots, t_s),$$

$m = 2^a$, $a > 1$ (i.e., $\mathbf{H}_{4m}$ is a 2-group) and $f: Q \to Q'$ as above. Let L_ξ be the characteristic class of n-plane bundles ξ defined in [MS, §7] and $L_Q = L_{\nu(Q)}$, where $\nu(Q)$ is the stable normal bundle of Q.

Lemma 1.2.2. *There exist quaternionic space-forms Q_1 and Q_1', 8-fold covered by Q and Q', respectively, and a polarization preserving homotopy equivalence $f_1: Q_1 \to Q_1'$ covered by f up to (based) homotopy, such that $f_1^*(L_{Q_1'}) = L_{Q_1}$.*

Proof. We construct $Q_2' = Q_{64m}(k + \alpha m, k + \beta m, t_1, \ldots, t_s)$ with α, β odd numbers, 16-fold covered by Q' and a polarized homotopy equivalence $f_2: Q_2 \to Q_2'$ covered by f (up to homotopy), where $Q_2 = Q_{64m}(k, k, t_1, \ldots, t_s)$. By Proposition (1.1.4), taking $Q_1 = Q_{32m}(k, k, t_1, \ldots, t_s)$ and $Q_1' = Q_{32m}(k + \alpha m, k + \beta m, t_1, \ldots, t_s)$, there is a polarized homotopy equivalence $f_1: Q_1 \to Q_1'$ such that the diagram

$$\begin{array}{ccc}
Q & \xrightarrow{f} & Q' \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1'} \\
Q_1 & \xrightarrow{f_1} & Q_1' \\
{\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi_2'} \\
Q_2 & \xrightarrow{f_2} & Q_2'
\end{array}$$

commutes up to homotopy, where the vertical arrows are the covering projections.

Q_2 and Q_2' are 2-fold covered by the lens spaces (in the notation of [M])

$$L = L_{32m}(q\,, -q\,, q\,, -q\,, r_1\,, -r_1\,, \dots\,, r_n\,, -r_n)$$

and

$$L' = L_{32m}(q'\,, -q'\,, q'\,, -q'\,, r_1\,, -r_1\,, \dots\,, r_n\,, -r_n)\,,$$

where $q.k \equiv 1 \pmod{32m}$, $q'.(k+m) \equiv 1 \pmod{32m}$ and $r_i.t_i \equiv 1 \pmod{32m}$, $i = 1\,, \dots\,, s$, for which, there is a homotopy commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\ g\ } & L' \\ \pi\big\downarrow & & \big\downarrow\pi' \\ Q_2 & \xrightarrow{\ f_2\ } & Q_2' \end{array}$$

where g is a homotopy equivalence compatible with the standard choice of generators of π_1. Then, we show that we can choose α and β so that $g^*(L_{L'}) = L_L$.

Since $\pi^*(f_2^*(L_{Q_2'}) - L_{Q_2}) = g^*(L_{L'}) - L_L = 0$ and

$$\pi^*\colon H^{4*}(Q_2\,, \mathbf{Z}_{(2)}) \to H^{4*}(L\,, \mathbf{Z}_{(2)})\,, \qquad \pi_2^*\colon H^{4*}(Q_2\,, \mathbf{Z}_{(2)}) \to H^{4*}(Q_1\,, \mathbf{Z}_{(2)})$$

have the same kernel, we get

$$f_1^*(L_{Q_1'}) - L_{Q_1} = \pi_2^*(f_2^*(L_{Q_2'}) - L_{Q_2}) = 0\,,$$

i.e., $f_1^*(L_{Q_1'}) = L_{Q_1}$.

By Proposition (1.1.4), the existence of f_2 is equivalent to solving

$$k^4 t_1^2 \cdots t_s^2 \equiv (k + \alpha m)^2 (k + \beta m)^2 t_1^2 \cdots t_s^2 \pmod{64m}$$

for α and β. This can be rewritten as

(1.2.3) $$2(\alpha + \beta)k + (\alpha^2 + \beta^2)m + 4\alpha\beta m \equiv 0 \pmod{64}.$$

To show that $g^*(L_{L'}) = L_L$, it suffices to prove that $g^*(p') = p$, where p and p' are the total Pontryagin classes of L and L', respectively (see [CS2, p. 326]). In [M, Sz], the Pontryagin classes of a lens space are described in terms of symmetric functions; following this description, $g^*(L_{L'}) = L_L$ can be reformulated as

$$\begin{aligned} &s_j(k^2\,, k^2\,, k^2\,, k^2\,, t_1^2\,, t_1^2\,, \dots\,, t_s^2\,, t_s^2) \\ &\quad \equiv s_j((k + \alpha m)^2\,, (k + \alpha m)^2\,, (k + \beta m)^2\,, (k + \beta m)^2\,, t_1^2\,, t_1^2\,, \dots\,, t_s^2\,, t_s^2) \\ &\qquad \pmod{32m} \end{aligned}$$

where s_j is the jth elementary symmetric function on $(2s+4)$ variables. This can be reduced to

(1.2.4) $$2(\alpha + \beta)k + (\alpha^2 + \beta^2)m \equiv 0 \pmod{16}$$

since $m = 2^a$, $a \geq 2$. Notice that it suffices to solve (1.2.3) since it implies equation (1.2.4). A solution to (1.2.3) is, for example, $\beta = 1$ and $\alpha = -1+m.t$, if $m = 2^a$, $a > 2$ or $\alpha = -1 + 20.t$, if $m = 4$, where $t.k \equiv 1 \pmod{64}$.

Proof of Proposition 1.2.1. By the discussion of section 1.1, f factors as $f = f_n \cdots f_1$, where

$$f_j\colon Q_{4m}(k_1 + m, k_1 + m, \ldots, k_{j-1} + m, k_{j-1} + m, \\ k_j, k_j, \ldots, k_n, k_n, t_1, \ldots, t_s) \\ \to Q_{4m}(k_1 + m, k_1 + m, \ldots, k_j + m, k_j + m, \\ k_{j+1} k_{j+1}, \ldots, k_n, k_n, t_1, \ldots, t_s)$$

$(j = 1, \ldots, n)$, is a polarization-preserving homotopy equivalence. Set $g_j = f_n.f_{n-1} \cdots f_{j+1}$ $(j = 1, \ldots, n-1)$. The composition formula for normal invariants yields

$$\eta(f) = \eta(f_n) + (g_{n-1}^*)^{-1}[\eta(f_{n-1})] + \cdots + (g_1^*)^{-1}[\eta(f_1)].$$

Therefore, we can assume that

$$Q = Q_{4m}(k, k, t_1, \ldots, t_s) \quad \text{and} \quad Q' = Q_{4m}(k + m, k + m, t_1, \ldots, t_s)$$

since a permutation of $(k_1, k_1, \ldots, k_n, k_n, t_1, \ldots, t_s)$ clearly gives (linearly) diffeomorphic polarized space-forms.

$[Q'; G/\mathrm{Top}]$ is a finite abelian group (see e.g. [CS2]), for G/Top carrying the H-space structure described in [Su]. To determine $\eta(f)_{\mathrm{odd}}$, the odd part of the normal invariant, we can pass to 2-fold covering spaces. Taking $\tilde{Q}$ ($\tilde{Q}'$, resp.) as the covering space of Q (Q', resp.) corresponding to the subgroup $\ker \psi_1$ ($\ker \psi_1'$, resp.) of $\pi_1(Q)$ ($\pi_1(Q')$, resp.), where ψ_1 (ψ_1', resp.) is the homomorphism defined in (1.1.2), it follows from Propositions (1.1.4) and (1.1.5) that $f\colon Q \to Q'$ is covered up to homotopy by a diffeomorphism $\tilde{f}\colon \tilde{Q} \to \tilde{Q}'$. Therefore, $\eta(f)_{\mathrm{odd}} = 0$.

Similarly, to determine $\eta(f)_{(2)} \in [Q'; (G/\mathrm{Top})_{(2)}]$, the even part of the normal invariant, we can take the largest odd-fold covering space and hence, assume that $m = 2^a$.

For $a = 1$, by Proposition (1.1.5), f is homotopic to a (linear) diffeomorphism, so that $\eta(f) = 0$. Therefore, it remains to show that $\eta(f)_{(2)} = 0$, when $m = 2^a$, $a \geq 2$.

G/Top has the homotopy type, upon localization at the prime 2, of a product of Eilenberg-Mac Lane spaces

$$(G/\mathrm{Top})_{(2)} \sim \prod_{i \geq 1} K(\mathbf{Z}_{(2)}, 4i) \times \prod_{i \geq 0} K(\mathbf{Z}_2, 4i + 2).$$

This description of $(G/\mathrm{Top})_{(2)}$ can be gotten by means of classes $L \in H^{4*}(G/\mathrm{Top}, \mathbf{Z}_{(2)})$ and $\kappa \in H^{4*+2}(G/\mathrm{Top}, \mathbf{Z}_2)$ obtained in [MS and RS]. Hence, we must prove that $\eta(f)^*(L) = 0$ and $\eta(f)^*(\kappa) = 0$. We start with $\eta(f)^*(L)$.

Let $f_1\colon Q_1 \to Q_1'$ be the map obtained in Lemma (1.2.2). In [MS] it is shown that

$$8.\eta(f_1)^*(L) + 1 = L_\xi . L_{Q_1'}^{-1},$$

where ξ is a bundle over Q_1' with $f_1^*(\xi) = \nu(Q_1)$. Therefore,

$$8.\eta(f_1)^*(L) = -1 + (f_1^*)^{-1}(L_{Q_1}).L_{Q_1'}^{-1} = -1 + L_{Q_1'}.L_{Q_1'}^{-1} = 0$$

i.e., $8.\eta(f_1)^*(L) = 0$. This implies that $\eta(f)^*(L) = \pi_1^*(\eta(f_1)^*(L)) = 0$, since

$$\eta(f_1)^*(L) \in \ker \pi_1^* = \{x \in H^{4*}(Q_1', \mathbf{Z}_{(2)}) | 8.x = 0\}.$$

As to $\eta(f)^*(\kappa)$, we first note that, since f is 2-fold covered by a diffeomorphism, $\eta(f)^*(\kappa) \in \ker \pi_3^*$, where $\pi_3^*\colon H^{4*+2}(Q', \mathbf{Z}_2) \to H^{4*+2}(\widetilde{Q}', \mathbf{Z}_2)$ is induced by the covering projection. On the other hand, by Lemma (1.2.2) (actually, a 2-fold cover would suffice) it follows that $\eta(f)^*(\kappa) \in \operatorname{im}(\pi_1')^*$.

Hence, $\eta(f)^*(\kappa) \in \ker \pi_3^* \cap \operatorname{im}(\pi_1')^* = 0$.

2. Browder-Livesay groups and multisignature invariants

In this section we first briefly review the definition of Browder-Livesay groups and the exact sequence relating them to L-groups following the discussion presented in [CS1]. Then, using multisignature invariants, we obtain some explicit computations needed for our geometric applications.

Let H be a finite group and

$$1 \to H \to \pi \xrightarrow{\psi} \{\pm 1\} \to 1$$

an extension of H by the multiplicative group $\{\pm 1\}$. Given $t \in \pi$ with $\psi(t) = -1$, we define $\beta_t(h) = th^{-1}t^{-1}$, $h \in H$ and extend it linearly to the integral group ring $\mathbf{Z}H$.

Browder-Livesay groups are the L-groups of the antistructure $(\mathbf{Z}H, \beta_t, t^2)$ [R, W] and will be denoted $L_n^e(\mathbf{Z}H, \beta_t, t^2)$. In general, the superscript $e = S$, s, h indicates that torsion is measured in $K_1(\mathbf{Z}H)$, $\mathrm{Wh}(H)$ and the trivial group, respectively.

A homomorphism $j\colon L_0^s(\mathbf{Z}H, \beta_t, t^2) \to L_0^s(\pi)$ is defined as follows.

Let $y \in L_0^s(\mathbf{Z}H, \beta_t, t^2)$ be represented by a quadratic module (M, q). Set $M_1 = M \otimes_{\mathbf{Z}H} \mathbf{Z}\pi$ and $q_1(x) = t^{-1}.q(x) \in \mathbf{Z}\pi$ for $x \in M = M \otimes 1 \subseteq M_1$; q_1 extends uniquely to $\mathbf{Z}\pi$ to yield a stably based unimodular quadratic form (M_1, q_1) over $(\mathbf{Z}\pi, \alpha, 1)$, $\alpha(g) = g^{-1}$ for $g \in \pi$. Define $j(y) = y_1$, where $y_1 \in L_0^s(\pi)$ is represented by (M_1, q_1). One can similarly define maps j^e, $e = S, h$.

Let $\psi!\colon L_n^s(\pi) \to L_1^s(\pi, H, \psi)$ be the homomorphism that geometrically corresponds to inducing line bundles on a degree 1 normal map with fundamental group π (the line bundle determined by $\psi\colon \pi \to \{\pm 1\}$).

Proposition 2.1. *The sequence*

$$L_0^s(\mathbf{Z}H, \beta_t, t^2) \xrightarrow{j} L_0^s(\pi) \xrightarrow{\psi!} L_1^s(\pi, H, \psi)$$

is exact.

For a proof, see [CS1].

Multisignature invariants. A multisignature invariant s: $L_0^S(\mathbf{Z}H, \beta_t, t^2) \to RO(H)$, where $RO(H)$ is the real representation ring of H is defined as follows. We first extend β_t to $\mathbf{R}H$, the real group ring of H, obtaining an antistructure $(\mathbf{R}H, \beta_t, t^2)$. Let $\mathbf{R}H = \prod_{i=1}^n S_i$ be the decomposition of the semisimple algebra $\mathbf{R}H$ as the product of its simple factors S_i. It is well known that the factors S_i are either invariant under β_t or interchanged in pairs; the latter, in the notation of Wall, are called factors of type GL (general linear) and give no contributions to L-groups. For the β_t-invariant simple factors we get antistructures (S_i, β_t^i, t_i^2), where β_t^i is the restriction of β_t to S_i and $t_i^2 = p_i(t^2)$, p_i: $\mathbf{R}H \to S_i$ being the projection onto the ith factor. The L-groups then split in the direct sum

$$L_j^S(\mathbf{R}H, \beta_t, t^2) = \sum_{i=1}^n L_j^S(S_i, \beta_t^i, t_i^2). \tag{2.2}$$

In [W2], antistructures (R, α, u) over simple rings are classified in type **O** (orthogonal), **Sp** (symplectic) and **U** (unitary), as follows:

L-groups of (R, α, u) are unchanged under Morita equivalences and therefore are the same for the antistructures (R, α, u), (R, α', u') and (R_n, α_n, u_n), where $u' = v[\alpha(v)]^{-1}u$, $\alpha'(x) = v\alpha(x)v^{-1}$, for some unit $v \in R$ and R_n is the ring of $n \times n$ matrices over R, u_n is the scalar matrix with u on the diagonal and α_n applies α to each entry and transpose. The isomorphism b between $L_0^s(R, \alpha, u)$ and $L_0^s(R, \alpha', u')$ is given by $b(M, q) = (M, q_1)$, where (M, q) is a quadratic module representing an element of $L_0^s(R, \alpha, u)$ and $q_1(x) = v.q(x)$.

Under the above equivalences, (R, α, u) can always be reduced to the case where R is a division ring and $u = \pm 1$; $(R, 1, -1)$ is the only case where the unit cannot be normalized to be 1. For R of finite dimension n^2 over its centre Z, (R, α, u) is said to be of type

U if $\alpha|Z$ is not the identity map.

O is $\alpha|Z$ is the identity map and the dimension of the fixed set of α is $(n^2 + un)/2$.

Sp if $\alpha|Z$ is the identity map and the dimension of the fixed set of α is $(n^2 - un)/2$.

Applying the above reductions to the simple summands (S_i, β_t^i, t_i^2), we can define a signature invariant I_i: $L_0^S(S_i, \beta_t^i, t_i^2) \to \mathbf{Z}$, by taking the signature of a representative symmetric quadratic form (after normalization) for the components where the unit can be normalized to 1, as in [W1]. For the components where we get -1 as the normalized unit, we define $I_i \equiv 0$.

The multisignature of $y \in L_0^S(\mathbf{Z}H, \beta_t, t^2)$ is the collection $\{I_i(y_i)\}$, where y_i is the ith component of $l(y)$ in the decomposition (2.2) and

$$l: L_0^S(\mathbf{Z}H, \beta_t, t^2) \to L_0^S(\mathbf{R}H, \beta_t, t^2)$$

is the homomorphism induced by inclusion. More precisely, if S_i is the simple factor of $RO(H)$ corresponding to the irreducible character χ_i, the multisignature of y is given by $s(y) = \sum_i I_i(y_i).\chi_i$.

We apply the above discussion to the antistructure $(\mathbf{ZH}_{2^{r+1}}, \beta_y, y^2)$ obtained from the extension

$$1 \to \mathbf{H}_{2^{r+1}} \to \mathbf{H}_{2^{r+2}} \xrightarrow{\psi} \{\pm 1\} \to 1,$$

where ψ is the homomorphism called ψ_1 in (1.1.2)(a). The antistructures over the simple factors of $\mathbf{RH}_{2^{r+1}}$ corresponding to the irreducible representations (in the notation of (1.1.2)) of $\mathbf{H}_{2^{r+1}}$ are:

$$\psi_0, \phi_0 - (\mathbf{R}, 1, 1) \quad \text{type } \mathbf{O},$$
$$\psi_1, \phi_1 - (\mathbf{R}, 1, -1) \quad \text{type } \mathbf{Sp}.$$

For δ_k we obtain a factor $\mathbf{R}_2$ (2×2 real matrices); taking $\nu = \beta_{-k}$ (the rotation by the angle $-\pi.k/m$, $m = 2^r$), we get the antistructure $(\mathbf{R}_2, t, 1)$, where t denotes transposition. After a further Morita equivalence we get $(\mathbf{R}, 1, 1)$, which is type $\mathbf{O}$. Similarly, we take for θ_k

$$\nu = \begin{bmatrix} \beta_{-k} & 0 \\ 0 & \beta_k \end{bmatrix}$$

to get $(\mathbf{H}, c, 1)$, which is type $\mathbf{Sp}$, where $\mathbf{H}$ is the division ring of the real quaternions and c is the standard conjugation in $\mathbf{H}$. It follows from the results of [W6] (section 5.2) that the image of the multisignature $s: L_0^S(\mathbf{ZH}_{2^{r+1}}, \beta_y, y^2) \to RO(\mathbf{H}_{2^{r+1}})$ contains the subgroup Λ_1 of $RO(\mathbf{H}_{2^{r+1}})$ generated by

$$\begin{array}{lll} & 2.\chi(\psi_0), & 8.\chi(\phi_0), \\ (2.3) & 8.\chi(\delta_k), & 0 < k < 2^{r-1},\ k \text{ even}, \\ & 2.\chi(\theta_k), & 0 < k < 2^{r-1},\ k \text{ odd}. \end{array}$$

Next, we describe j^S on a representation ring level. Letting $j_*: RO(\mathbf{H}_{2^{r+1}}) \to RO(\mathbf{H}_{2^{r+2}})$ be the homomorphism given on irreducible characters by

$$j_*(\chi(\psi_0')) = \chi(\psi_0) - \chi(\psi_1),$$
$$j_*(\chi(\varphi_0')) = \chi(\phi_0) - \chi(\phi_1),$$
$$j_*(\chi(\psi_1')) = j_*(\chi(\phi_1')) = 0,$$
$$j_*(\chi(\delta_k')) = \chi(\delta_k) - \chi(\delta_{m-k}),$$
$$j_*(\chi(\varphi_k')) = \chi(\theta_k) - \chi(\theta_{m-k}), \qquad m = 2^r,$$

where $'$ is used to distinguish the representations of $\mathbf{H}_{2^{r+1}}$ and $\mathbf{H}_{2^{r+2}}$, it is not hard to see from the definition of j^S, that the diagram

$$\text{(2.4)}\qquad \begin{array}{ccc} L_0^S(\mathbf{ZH}_{2^{r+1}}, \beta_y, y^2) & \xrightarrow{j^S} & L_0^S(\mathbf{H}_{2^{r+2}}) \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \bar{s}} \\ RO(\mathbf{H}_{2^{r+1}}) & \xrightarrow{j_*} & RO(\mathbf{H}_{2^{r+2}}) \end{array}$$

commutes. Here, $\bar{s}$ is the multisignature as defined in [W3].

Letting Λ be the subgroup of $RO(\mathbf{H}_{2^{r+2}})$ generated by

$$\text{(2.5)}\qquad \begin{array}{ll} 2.[\chi(\psi_0) - \chi(\psi_1)], & 8.[\chi(\phi_0) - \chi(\phi_1)], \\ 8.[\chi(\delta_k) - \chi(\delta_{m-k})], & 0 < k < m,\ k \text{ even}, \\ 2.[\chi(\theta_k) - \chi(\theta_{m-k})], & 0 < k < m,\ k \text{ odd}, \end{array}$$

it follows from (2.3) and (2.4) that

$$\text{(2.6)}\qquad \Lambda \subseteq \bar{s}.j^S(L_0^S(\mathbf{ZH}_{2^{r+1}}, \beta_y, y^2)).$$

Let $t: L_0^s(\mathbf{H}_{2^{r+2}}) \to L_0^s(\mathbf{H}_{2^{r+1}})$ be the composite

$$L_0^s(\mathbf{H}_{2^{r+2}}) \xrightarrow{\psi!} L_1^s(\mathbf{H}_{2^{r+2}}, \mathbf{H}_{2^{r+1}}, \psi) \xrightarrow{\partial} L_0^s(\mathbf{H}_{2^{r+1}}),$$

where ∂ is the boundary operator in the relative surgery sequence.

Proposition 2.7. $\ker t = \ker \psi!$

Proof. By Theorem (5.3.4) of [W6] and taking into account that $SK_1(\mathbf{ZH}_{2^{r+2}}) = 0$ [O], it follows that the multisignature s_1 (s_2, resp.) maps $L_0^s(\mathbf{H}_{2^{r+2}})$ ($L_0^s(\mathbf{H}_{2^{r+1}})$, resp.) isomorphically onto Σ_1 (Σ_2, resp.), where

$$\Sigma_1 = \bigoplus\{8.\mathbf{Z},\ \text{type } \mathbf{O} \text{ factors of } \mathbf{RH}_{2^{r+2}}\} \bigoplus\{2.\mathbf{Z},\ \text{type } \mathbf{Sp} \text{ factors of } \mathbf{RH}_{2^{r+2}}\}$$

(similarly for Σ_2).

Letting $\mathbf{R}: RO(\mathbf{H}_{2^{r+2}}) \to RO(\mathbf{H}_{2^{r+1}})$ be the restriction homomorphism, i.e., restricts a character χ to the subgroup $\mathbf{H}_{2^{r+1}}$, we get the commutative diagram

$$\begin{array}{ccc} L_0^S(\mathbf{H}_{2^{r+2}}) & \xrightarrow{t} & L_0^s(\mathbf{H}_{2^{r+1}}) \\ {\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle s_2} \\ \Sigma_1 & \xrightarrow{r} & \Sigma_2 \end{array}$$

where r is the restriction of R to Σ_1.

A simple calculation shows that $\ker r = \Lambda$, where Λ is the subgroup of Σ_1 described in (2.5). Hence, $\ker t = s_1^{-1}(\Lambda)$.

By (2.6) and the commutativity of

$$\begin{array}{ccc}
L_0^S(\mathbf{ZH}_{2^{r+1}}, \beta_y, y^2) & \longrightarrow & L_0^s(\mathbf{ZH}_{2^{r+1}}, \beta_y, y^2) \\
\big\downarrow{\scriptstyle j^S} & & \big\downarrow{\scriptstyle j} \\
L_0^S(\mathbf{H}_{2^{r+2}}) & \longrightarrow & L_0^s(\mathbf{H}_{2^{r+2}}) \\
\big\downarrow{\scriptstyle \bar{s}} & & \big\downarrow{\scriptstyle s_1} \\
RO(\mathbf{H}_{2^{r+2}}) & \xrightarrow{\text{id}} & RO(\mathbf{H}_{2^{r+2}})
\end{array}$$

we conclude that $\ker t = s_1^{-1}(\Lambda) \subseteq \operatorname{im}(j) = \ker \psi!$.

The inclusion $\ker \psi! \subseteq \ker t$ follows from $t = \partial . \psi!$.

3. The main theorem

In this section, we construct new examples of nonlinearly equivalent representations of the quaternionic groups $\mathbf{H}_{2^{r+2}}$, $r \geq 2$.

Let k be an odd number and $m = 2^r$.

Theorem 3.1. *The representations $4\theta_k + \psi_1$ and $4\theta_{k+m} + \psi_1$ of $\mathbf{H}_{2^{r+2}}$ are topologically equivalent.*

Let $Q_k = Q_{4m}(k_1, \dots, k_n)$ and $Q_{k+m} = Q_{4m}(k+m, k+m, k+m, k+m)$. By Proposition (1.2.1), there is a topological normal cobordism (V, Q_k, Q_{k+m}) compatible with polarizations, i.e., a degree one normal map

$$g: (V, Q_k, Q_{k+m}) \to (Q_{k+m} \times I, Q_{k+m}, Q_{k+m})$$

such that $g|_{Q_{k+m}}$ is the identity map of Q_{k+m} and $f = g|_{Q_k} : Q_k \to Q_{k+m}$ is a polarization preserving homotopy equivalence.

Let E_{k+m} be the total space of the $[-1, 1]$-bundle over Q_{k+m} with first Stiefel-Whitney class represented by $\psi : \pi_1(Q_{k+m}) \to \{\pm 1\}$, the composite of the polarization isomorphism and ψ_1. Let

$$\bar{g}: (E, E_k, E_{k+m}) \to (E_{k+m} \times I, E_{k+m}, E_{k+m}) \tag{3.2}$$

be the surgery problem obtained from g by inducing line bundles; E, E_k are the total spaces of the induced line bundles over V and Q_k, respectively.

The polarization of Q_k induces an identification $\pi_1(E_k) \approx \mathbf{H}_{2^{r+2}}$; we shall call its outer isomorphism class, the *preferred generators* of $\pi_1(E_k)$.

Proposition 3.3. *There is a topological h-corbordism (W, E_k, E_{k+m}) respecting preferred generators of π_1.*

Proof. It suffices to show that the surgery obstruction $\sigma(\bar{g}) = \psi!(\sigma(g))$ of the surgery problem (3.2) described above is zero in the relative surgery group $L_1^h(\mathbf{H}_{2^{r+2}}, \mathbf{H}_{2^{r+1}}, \psi)$, which will be denoted $L_1(\mathbf{H}_{2^{r+2}}^-, \mathbf{H}_{2^{r+1}})$. By Lemma (3.4) below, it follows that $\sigma(g) = \gamma(x)$ for some $x \in L_0^s(\mathbf{H}_{2^{r+2}})$, where $\gamma: L_0^s(\mathbf{H}_{2^{r+2}})$

$\to L_0^h(\mathbf{H}_{2^{r+2}})$ is the forgetful map (forgets preferred basis). By Proposition (2.7) and the commutativity of the diagram

$$\begin{array}{ccccc} L_0^s(\mathbf{H}_{2^{r+2}}) & \xrightarrow{\psi!} & L_1^s(\mathbf{H}_{2^{r+2}}^-, \mathbf{H}_{2^{r+1}}) & \xrightarrow{\partial} & L_0^s(\mathbf{H}_{2^{r+1}}) \\ \downarrow \gamma & & \downarrow \gamma & & \downarrow \gamma \\ L_0^h(\mathbf{H}_{2^{r+2}}) & \xrightarrow{\psi!} & L_1(\mathbf{H}_{2^{r+2}}^-, \mathbf{H}_{2^{r+1}}) & \xrightarrow{\partial} & L_0^h(\mathbf{H}_{2^{r+1}}) \end{array}$$

it suffices to show that $t(x) = 0$, where $t = \partial.\psi!$ is the transfer to the 2-fold cover ($\psi!$ in the context of homotopy equivalences is defined as in §2 except that one does not keep track of basis).

According to [W6], $L_0^s(\mathbf{H}_{2^{r+1}})$ is completely determined by the signature invariant $I: L_0^s(\mathbf{H}_{2^{r+1}}) \to L_0^s(e)$ and the multisignature to the reduced representation ring of $\mathbf{H}_{2^{r+1}}$; therefore, we must show that they vanish for $t(x)$. Since these invariants are also defined in $L_0^h(\mathbf{H}_{2^{r+1}})$, it is sufficient to check that they vanish for $t(\sigma(g)) \in L_0^h(\mathbf{H}_{2^{r+1}})$. Geometrically, they are the signature and multisignature of the normal cobordism $\widetilde{V}$ between the 15-dimensional manifolds $\widetilde{Q}_k = Q_{2m}(k, k, k, k)$ and $\widetilde{Q}_{k+m} = \widetilde{Q}_{2m}(k+m, k+m, k+m, k+m)$, obtained on the boundary of E ($\widetilde{Q}_k$ and $\widetilde{Q}_{k+m}$ are the boundaries of E_k and E_{k+m}, respectively). Clearly, $\widetilde{V}$ is a 2-fold covering space of V.

We can assume that the signature of V is zero after taking, if necessary, the connected sum with a Milnor manifold (see e.g. [Br]). By Proposition (1.1.5), the space-forms Q_k and Q_{k+m} on the boundary of V are homeomorphic. Let V' be the closed manifold obtained from V by identifying Q_k and Q_{k+m} under the above homeomorphism. A similar construction for $\widetilde{V}$ yields a manifold $\widetilde{V}'$, which double covers V'. As quaternionic space-forms are rational homology spheres, letting I denote the signature, we get $I(V) = I(V')$ and $I(\widetilde{V}') = I(\widetilde{V}')$. By the Hirzebruch signature theorem, I is multiplicative for covering spaces of closed manifolds and therefore,

$$I(\widetilde{V}) = I(\widetilde{V}') = 2.I(V') = 2.I(V) = 0,$$

since $I(V) = 0$.

The multisignature of $\widetilde{V}$ is detected by the difference of the ρ-invariants of the boundary components $\widetilde{Q}_k$ and $\widetilde{Q}_{k+m}$ of $\widetilde{V}$ [W3]. Since $\widetilde{Q}_k$ and $\widetilde{Q}_{k+m}$ are the same polarized quaternionic space-form, this difference is zero. This concludes the proof.

Lemma 3.4. $\sigma(g) = \gamma(x)$ *for some* $x \in L_0^s(\mathbf{H}_{2^{r+2}})$, *where*

$$\gamma: L_0^s(\mathbf{H}_{2^{r+2}}) \to L_0^h(\mathbf{H}_{2^{r+2}})$$

is the forgetful map.

Proof. By the Rothenberg sequence [Sh]

$$L_0^s(\mathbf{H}_{2^{r+2}}) \xrightarrow{\gamma} L_0^h(\mathbf{H}_{2^{r+2}}) \xrightarrow{\beta} H^0(\mathbf{Z}_2, \mathrm{Wh}(\mathbf{H}_{2^{r+2}})),$$

where $H^i(\mathbf{Z}_2, \mathrm{Wh}(\mathbf{H}_{2^{r+2}}))$ is the ith cohomology group of the $\mathbf{Z}_2$ action on $\mathrm{WH}(\mathbf{H}_{2^{r+2}})$ induced by the involution $\iota: \mathbf{ZH}_{2^{r+2}} \to \mathbf{ZH}_{2^{r+2}}$, given on group elements by $\iota(g) = g^{-1}$, it suffices to show that $\beta(\sigma(g)) = 0$.

In [M], a torsion $\tau \in \mathrm{Wh}(\mathbf{Q}\pi/(\Sigma))$ is defined for spherical space-forms, i.e., the orbit space of a free action of the finite group π on a sphere. Here, $\mathbf{Q}\pi$ is the rational group ring of π and (Σ) is the ideal generated by $\Sigma = \sum_{g\in\pi} g$. From this, we get a Reidemeister torsion $\tau_R = \mathrm{Nrd}(\tau)$, where $\mathrm{Nrd}: \mathrm{Wh}(\mathbf{Q}\pi/(\Sigma)) \to \mathbf{U}(\mathbf{Q}\pi/(\Sigma))$ is the reduced norm (see [B] for a definition) and $\mathbf{U}$ stands for units in the centre of $\mathbf{Q}\pi/(\Sigma)$.

Let $\alpha: \mathrm{Wh}(\mathbf{ZH}_{2^{r+2}}) \to \mathbf{U}(\mathbf{QH}_{2^{r+2}}/(\Sigma))$ be the composite

$$\mathrm{Wh}(\mathbf{ZH}_{2^{r+2}}) \xrightarrow{i} \mathrm{Wh}(\mathbf{QH}_{2^{r+2}}) \xrightarrow{\mathrm{Nrd}} \mathbf{U}(\mathbf{QH}_{2^{r+2}}) \longrightarrow \mathbf{U}(\mathbf{QH}_{2^{r+2}}/(\Sigma)),$$

which is injective since $SK_1(\mathbf{H}_{2^{r+2}}) = \ker i = 0$ and Nrd is injective for semisimple rings over $\mathbf{Q}$ [We].

$\beta(\sigma(g))$ is represented by the torsion $\tau \in \mathrm{Wh}(\mathbf{H}_{2^{r+2}})$ of the homotopy equivalence $f: Q_k \to Q_{k+m}$. Then, $\alpha(\tau) = \tau_R(Q_{k+m}).\tau_R^{-1}(Q_k)$, the quotient of the Reidemeister torsions of the space-forms on the boundary of the normal cobordism V (see e.g. [W3]). By [W5], $\mathrm{SK}_1(\mathbf{H}_{2^{r+2}})$ is the torsion subgroup of $\mathrm{Wh}(\mathbf{H}_{2^{r+2}})$; as $\mathrm{SK}_1(\mathbf{H}_{2^{r+2}}) = 0$, it follows that $\mathrm{Wh}(\mathbf{H}_{2^{r+2}})$ is free and hence, $\mathbf{Z}_2$ acts trivially on it. Therefore, $H^0(\mathbf{Z}_2, \mathrm{Wh}(\mathbf{H}_{2^{r+2}})) = \mathrm{Wh}(\mathbf{H}_{2^{r+2}})/2\cdot\mathrm{Wh}(\mathbf{H}_{2^{r+2}})$, and it suffices to show that $\alpha(\tau) = (\alpha(\bar\tau))^2$ for some $\bar\tau \in \mathrm{Wh}(\mathbf{H}_{2^{r+2}})$.

Let $\bar f: Q_{4m}(k, k) \to Q_{4m}(k+m, k+m)$ be the polarized homotopy equivalence of 7-dimensional quaternionic space-forms obtained in Proposition (1.1.4) and $\bar\tau = \tau(\bar f)$ its torsion. By the above considerations

$$\alpha(\bar\tau) = \tau_R(Q_{4m}(k+m, k+m)).\tau_R^{-1}(Q_{4m}(k, k))$$

and one can see from the periodicity of the cellular chain complexes of quaternionic space-forms of the type $Q_{4m}(k, k, k, k)$ that $\tau_k(Q_k) = \tau_R^2(Q_{4m}(k, k))$ and $\tau_R(Q_{k+m}) = \tau_R^2(Q_{4m}(k+m, k+m))$. Therefore, $\alpha(\tau) = (\alpha(\bar\tau))^2$ and since α is injective, $\tau = 2\bar\tau$.

Proof of Theorem 3.1. Let (W, E_k, E_{k+m}) be the h-cobordism obtained in Proposition (3.3) and $\widetilde{W}$ the h-cobordism between $\widetilde{Q}_k$ and $\widetilde{Q}_{k+m}$ on the boundary of W. $\widetilde{Q}_k$ and $\widetilde{Q}_{k+m}$ are double covers of Q_k and Q_{k+m}, respectively.

Let $\tilde f: \widetilde{Q}_{k+m} \to \widetilde{Q}_k$ be the outer polarization preserving homotopy equivalence induced by $(\widetilde{W}, \widetilde{Q}_k, \widetilde{Q}_{k+m})$. Since $\widetilde{Q}_k$ and $\widetilde{Q}_{k+m}$ are the same polarized space-form, $\tau(\tilde f) = 0$. On the other hand, $\tau(\tilde f) = \tau(\widetilde{W}, \widetilde{Q}_k) - \tau(\widetilde{W}, \widetilde{Q}_{k+m})$. By the duality theorem for h-cobordisms [M], and since $\widetilde{Q}_k$ and $\widetilde{Q}_{k+m}$ are odd-dimensional manifolds, it follows that $\tau(\widetilde{W}, \widetilde{Q}_{k+m}) = -\tau^*(\widetilde{W}, \widetilde{Q}_k)$, where $*$ denotes the involution on $\mathrm{Wh}(\mathbf{H}_{2^{r+2}})$. As $\mathrm{Wh}(\mathbf{H}_{2^{r+2}})$ is torsion-free, $\tau^*(\widetilde{W}, \widetilde{Q}_k) =$

$\tau(\widetilde{W}, \tilde{Q}_k)$; therefore,

$0 = \tau(\tilde{f}) = \tau(\widetilde{W}, \tilde{Q}_k) - \tau(\widetilde{W}, \tilde{Q}_{k+m}) = \tau(\widetilde{W}, \tilde{Q}_k) - (-\tau(\widetilde{W}, \tilde{Q}_k)) = 2.\tau(\widetilde{W}, \tilde{Q}_k)$.

Hence, $\tau(\widetilde{W}, \tilde{Q}_k) = 0$. By the s-cobordism theorem, there is a homeomorphism $\tilde{h}_0: (\widetilde{W}, \tilde{Q}_k, \tilde{Q}_{k+m}) \to (\tilde{Q}_k \times I, \tilde{Q}_k, \tilde{Q}_k)$ with $\tilde{h}_0(x) = (x, 0)$, if $x \in \tilde{Q}_k$.

We decompose the unit sphere $\mathbf{S}^{16}$ of the representation space of $4\theta_k + \psi_1$ ($4\theta_{k+m} + \psi_1$, resp.) as the union

$$\mathbf{S}^{16} = \mathbf{D}^{16} \times \{\pm 1\} \cup \mathbf{S}^{15} \times [-1, 1],$$

with the two components identified along $\mathbf{S}^{15} \times \{\pm 1\}$, each of them being invariant under $4\theta_k + \psi_1$ ($4\theta_{k+m} + \psi_1$, resp.). From this, we get the decomposition

$$\mathbf{R}^{17} - \{0\} = \mathbf{D}^{16} \times \{\pm 1\} \times \mathbf{R} \cup \mathbf{S}^{15} \times [-1, 1] \times \mathbf{R}.$$

As taking product with S^1 kills torsion, $(W \times S^1, E_k \times S^1, E_{k+m} \times S^1)$ is a s-cobordism. Therefore, there is a homeomorphism

$$h_1: (W \times S^1, E_k \times S^1, E_{k+m} \times S^1) \to (E_k \times S^1 \times I, E_k \times S^1, E_k \times S^1)$$

with $h_1(x, t) = (x, t, 0)$, for $x \in E_k$, $t \in S^1$ and $h_1(x, t) = (\tilde{h}_0(x), t)$, for $x \in \widetilde{W}$, $t \in S^1$.

Since $\mathbf{S}^{15} \times [-1, 1] \times \mathbf{R}$ is the universal covering space of $E_k \times S^1$ ($E_{k+m} \times S^1$, resp.), we can lift $h_2 = h_1 | E_{k+m} \times S^1$ to obtain an equivariant map

$$h_3: \mathbf{S}^{15} \times [-1, 1] \times \mathbf{R} \to \mathbf{S}^{15} \times [-1, 1] \times \mathbf{R}$$

with $h_3(x, u, t) = (h_0, u, t)$ for any $x \in \mathbf{S}^{15}$, $u \in \{\pm 1\}$, $t \in \mathbf{R}$, for some equivariant homeomorphism $h_0: \mathbf{S}^{15} \to \mathbf{S}^{15}$. This is possible by the uniqueness of lifts to covering spaces (h_0 is a lift of $\tilde{h}_0$) and because the action on the radial coordinate t is trivial.

We extend h_0 radially to get an equivariant homeomorphism $h_0: \mathbf{D}^{16} \to \mathbf{D}^{16}$, and define

$$h_0: \mathbf{D}^{16} \times \{\pm 1\} \times \mathbf{R} \to \mathbf{D}^{16} \times \{\pm 1\} \times \mathbf{R}$$

by $h_4(x, u, t) = (h_0(x), u, t)$.

Since h_3 and h_4 agree on $\mathbf{S}^{15} \times \{\pm 1\} \times \mathbf{R}$, they yield an equivariant homeomorhism $h: \mathbf{R}^{17} - \{0\} \to \mathbf{R}^{17} - \{0\}$. Defining $h(0) = 0$, we get a homeomorphism $h: \mathbf{R}^{17} \to \mathbf{R}^{17}$, which conjugates $4\theta_k + \psi_1$ and $4\theta_{k+m} + \psi_1$. This concludes the proof.

Let η_1 and η_2 be finite-dimensional real linear representations of $\mathbf{H}_{2^{r+2}}$ of the form

$$\eta_i = \sum_{\substack{k=1 \\ k\,\text{odd}}}^{m-1} a_k^i, \theta_k + \psi_1 + \beta, \qquad i = 1, 2,$$

with a_k^i positive integers, $m = 2^r$.

Corollary 3.5. *η_1 and η_2 are topologically equivalent if $a_k^1 + a_{m-k}^1 = a_k^2 + a_{m-k}^2$ and $a_k^1 - a_k^2 \equiv 0 \pmod 4$, for any $0 < k < m/2$, k odd.*

References

[B] H. Bass, *Algebraic K-theory*, Benjamin, 1968.

[Br] W. Browder, *Surgery on simply-connected manifolds*, Springer-Verlag, 1972.

[CS1] S. E. Cappell and J. L. Shaneson, *Pseudo-free actions*. I, Lecture Notes in Math., vol. 763, Springer, 1979, pp. 395–447.

[CS2] ——, *Non-linear similarity*, Ann. of Math. **113** (1981), 315–355.

[CS3] ——, *Topological rationality of linear representations*, Publ. Inst. Hautes Etudes Sci. **66** (1983) 309–336.

[CS4] ——, *Linear similarity and non-linear similarity are equivalent below dimension* 6.

[CSSW] S. E. Cappell, J. L. Shaneson, M. Steinberger and J. E. West, *Nonlinear similarity begins in dimension six*, preprint.

[Co] M. M. Cohen, *A course in simple-homotopy theory*, Springer-Verlag, New York, 1973.

[Ma] I. Madsen, *Smooth spherical space forms*, Lecture Notes in Math., vol. 657, Springer, 1978, 303–352.

[M] J. Milnor, *Whitehead torsion*, Bull. Amer. Math. Soc. **72** (1966), 358–426.

[Mi] W. Mio, Thesis, New York Univ., 1984.

[MS] J. Morgan and D. Sullivan, *The transversality characteristic class and linking cycles in surgery theory*, Ann. of Math. **99** (1974), 463–544.

[O] R. Oliver, SK_1 *for finite group rings*. III, Lecture Notes in Math., vol. 854, Springer, 1980, pp. 299–337.

[R] A. A. Ranicki, *Algebraic L-theory*. I. *Foundations*, Proc. London Math. Soc. **27** (1973), 101–125.

[RS] C. P. Rourke and D. P. Sullivan, *On the Kervaire obstruction*, Ann. of Math. **94** (1971), 397–413.

[Se] J. P. Serre, *Representations linéaires des groupes finis*, Hermann, Paris, 1971.

[Sh] J. L. Shaneson, *Wall's surgery obstruction groups for* $Z \times G$, Ann. of Math. **90** (1969), 296–334.

[Su] D. Sullivan, *Triangulating and smoothing homotopy equivalences and homeomorphisms*, Geometric Topology Seminar Notes, Princeton Univ., 1967.

[Sz] R. H. Szczarba, *On tangent bundles of fiber spaces and quotient spaces*, Amer. J. Math. **86** (1964), 685–697.

[W1] C. T. C. Wall, *Surgery on non-simply connected manifolds*, Ann. of Math. **84** (1966), 217–276.

[W2] ——, *On the classification of Hermitian forms*. II. *Semisimple rings*, Invent. Math. **18** (1972), 119–141.

[W3] ——, *Surgery on compact manifolds*, Academic Press, 1970.

[W4] ——, *On the axiomatic foundations of the theory of Hermitian forms*, Proc. Cambridge Philos. Soc. **67** (1970), 243–250.

[W5] ——, *Norms of units in group rings*, Proc. London Math. Soc. **29** (1974), 593–632.

[W6] ——, *Classification of Hermitian forms.* VI: *Group rings*, Ann. of Math. **103** (1976), 1–80.

[W7] ——, *Free actions of finite groups on spheres*, Proc. Sympos. Pure Math., vol. 32, Amer. Math. Soc., Providence, R.I., 1978, pp. 115–124.

[We] A. Weil, *Basic number theory*, Springer-Verlag, New York, 1967.

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